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Noncommutative algebras for hyperbolic diffeomorphisms

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Summary. The Gibbs states of classical equilibrium statistical mechanics can be extended to states on non commutative algebras, satisfying the Kubo-Martin-Schwinger boundary condition. This way of looking at Gibbs states is applied here to the study of differentiable dynamical systems when some (strong or weak) hyperbolicity conditions are satisfied.

1. Introduction

The general study of differentiable dynamical systems (in particular their ergodic theory) is difficult, and detailed results are rare. For the special class of hyperbolic systems however (Anosov systems and more generally Axiom A systems) many results have been obtained following the construction of Markov partitions by Sinai [21], [22] and its improvement by Bowen [1]. Markov partitions permit the replacement of the original differentiable dynamics by *symbolic dynamics*, and ergodic problems on a manifold are replaced by problems of equilibrium statistical mechanics (on a one-dimensional lattice) for which one has effective methods (see [2], [15]). Unfortunately, the construction of Markov partitions is not canonical and, a priori, mathematical objects constructed with the help of a Markov partition are also not canonical.

In the present paper we define and discuss certain noncommutative algebras naturally associated with hyperbolic diffeomorphisms. (Hyperbolic flows could presumably be handled in similar manner, but will not be discussed here). The noncommutative algebras in question are of a general type introduced by Connes in connection with foliations [6]. Using these algebras one can make definitions which are manifestly independent of the choice of a Markov partition. (But note that Markov partitions remain important in making proofs).

Before embarking in the discussion of hyperbolic diffeomorphisms, it is convenient to analyze (in Section 2) a general definition of Gibbs states (Capocaccia [5]) and to show how it is naturally expressed in terms of a suitable C^*-algebra:

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Gibbs states ρ correspond to states $\hat{\rho}$ on the algebra, which are KMS for an explicitly given modular group of automorphisms.

Next, we discuss Smale spaces (Section 3), which permit the analysis of hyperbolic diffeomorphisms at a suitable level of abstraction. Using in particular a result of Haydn, one shows that different definitions of Gibbs states on a Smale space (using or not a Markov partition) are in fact equivalent. Some extensions (s- and u-Gibbs states, Gibbs distributions) are briefly discussed.

In Section 4, hyperbolicity is weakened to the requirement of having an ergodic measure with no zero characteristic exponent. It is still possible to define Gibbs states ρ in this framework: each one corresponds to a state $\hat{\rho}$ on a von Neumann algebra, and $\hat{\rho}$ satisfies the KMS condition with respect to an explicitly given modular group of automorphisms. Some examples are discussed, in particular that of SRB measures.

2. Reformulation of a definition of Gibbs states

We consider a compact metrizable space Ω and a representation $k \mapsto \tau^k$ of a countable infinite group Γ by homeomorphisms of Ω (the unit element of Γ is represented by the identity map of Ω). We assume that this representation is *expansive*; this means that for a given metric d compatible with the topology there is $\varepsilon > 0$ such that

$$d(\tau^k x, \tau^k y) \leqq \varepsilon \quad \text{for all } k \in \Gamma$$

implies $x = y$. We say that x and y are *conjugate* if

$$d(\tau^k x, \tau^k y) \to 0 \quad \text{when} \quad k \to \infty$$

Conjugacy is an equivalence relation, and the equivalence classes are countable. We assume that the following condition is satisfied.

(C) For every conjugate pair (x, y) there is a map φ: $\mathcal{O} \mapsto \Omega$ such that $\mathcal{O}$ is a neighborhood of x in Ω, φ is continuous at x, $\varphi(x) = y$, and

$$\lim_{k \to \infty} d(\tau^k z, \tau^k \varphi z) = 0 \tag{2.1}$$

uniformly for $z \in \mathcal{O}$.

Capocaccia [5] has remarked that if (*C*) holds, the germ of φ at x is uniquely determined by (x, y), and is a germ of homeomorphism.[2] It is thus natural to define a *conjugating homeomorphism* as a pair $(\mathcal{O}, \varphi)$ where $\mathcal{O}$ is an open subset of Ω, and φ is a homeomorphism of $\mathcal{O}$ to $\varphi\mathcal{O}$ such that (2.1) holds uniformly for $z \in \mathcal{O}$. The conjugating homeomorphisms form a pseudogroup of topological transformations of Ω.

It is convenient to introduce at this point a topological space G analogous to the "graph" of a foliation constructed by Winkelnkemper [26]. The points

[2] In [5] it is assumed that $\Gamma = \mathbb{Z}^\nu$ (the situation for statistical mechanics on a lattice), but the result extends to any countable infinite group Γ

of G are the conjugate pairs (x, y), and a base of the topology is given by the open sets

$$\{(z, \varphi(z)) : z \in \mathcal{O} \text{ and } (\mathcal{O}, \varphi) \text{ is a conjugating homeomorphism}\}$$

This topology is Hausdorff because the germ of φ is uniquely determined by (x, y), and it is clear that G is locally compact with countable base.

Let now V: $G \mapsto \mathbb{R}$ be continuous and such that $V(x, y) + V(y, z) = V(x, z)$. (In particular $V(x, x) = 0$ and $V(x, y) = -V(y, x)$). We say that a probability measure σ on Ω is a *Gibbs state* with respect to V(see [5]) if

$$\varphi[g_\varphi \cdot (\sigma | \mathcal{O})] = \sigma | \varphi\mathcal{O}$$

where

$$g_\varphi(x) = \exp - V(x, \varphi(x))$$

for every conjugating homeomorphism $(\mathcal{O}, \varphi)$. This means that the image by φ of σ restricted $\mathcal{O}$, up to multiplication by g_φ, is σ restricted to $\varphi\mathcal{O}$. There are many possible variations on this definition, but as presented it is a direct generalization of the definition for classical lattice spin systems, and appropriate for the Smale spaces to be studied in the next section.

We come now to the announced construction of a noncommutative algebra, along the lines of Conne's construction of the algebra associated with a foliation (see [6], Section 5). Let $\mathscr{C}_c(G)$ be the linear space of complex continuous functions with compact support on G. If A, $B \in \mathscr{C}_c(G)$ we define the product $A * B$ by

$$(A * B)(x, y) = \sum_z A(x, z) B(z,y)$$

where the sum is over all z which are conjugate to x and y. There are finitely many nonzero terms in the sum, and $A * B \in \mathscr{C}_c(G)$ as one checks readily, so that $\mathscr{C}_c(G)$ becomes an associative complex algebra. An involution $A \mapsto A^*$ is defined by

$$A^*(x, y) = \overline{A(y, x)}$$

where the bar denotes complex conjugation.

For each equivalence class $[x]$ of conjugate points of Ω there is a representation $\pi_{[x]}$ of the $*$-algebra $\mathscr{C}_c(G)$ in the Hilbert space $l^2([x])$ of square summable functions $[x] \to \mathbb{C}$, such that

$$((\pi_{[x]} A) \xi)(y) = \sum_{z \in [x]} A(y, z) \xi(z)$$

for $\xi \in l^2([x])$. Denoting by $\|\pi_{[x]} A\|$ the operator norm, we write

$$\|A\| = \sup_{[x]} \|\pi_{[x]} A\|$$

The completion of $\mathscr{C}_c(G)$ with respect to this norm is a separable C^*-algebra which we denote by $C^*(G)$.

If $A \in \mathscr{C}_c(G)$ and $t \in \mathbb{R}$, we write

$$(\sigma^t A)(x, y) = e^{iV(x,y)t} A(x, y)$$

defining a one-parameter group (σ^t) of $*$-automorphisms of $\mathscr{C}_c(G)$ and, by unique extension, a one-parameter group of $*$-automorphisms of $C^*(G)$.

A *state* α on $C^*(G)$ is a linear functional such that $\alpha(A * A) \geqq 0$ and $\alpha(1) = 1$. It is *invariant* if $\alpha \circ \sigma^t = \alpha$. An invariant state satisfies the *KMS boundary condition*[3] if for all $A, B \in C^*(G)$, there is a continuous function F on $\{z \in \mathbb{C}\colon\ 0 \leqq Im z \leqq 1\}$, holomorphic in $\{z \in \mathbb{C}\colon\ 0 < Im z < 1\}$, and such that

$$\alpha(\sigma_t A \cdot B) = F(t), \qquad \alpha(B \cdot \sigma_t A) = F(t+i)$$

[Note that it is sufficient to verify these various conditions on $\mathscr{C}_c(G)$].

2.1 Theorem (i) *If α is a probability measure on Ω then a state $\hat{\alpha}$ on $C^*(G)$ is defined by*

$$\hat{\alpha}(A) = \int \alpha(dx)\, A(x, x) \tag{2.2}$$

(ii) *$\hat{\alpha}$ satisfies the* KMS *boundary condition with respect to (σ^t) if and only if α is a Gibbs state with respect to V.*[4]

Concerning (i) we note that

$$|A(x, y)| \leqq \Big(\sum_{x \in [y]} |A(x, y)|^2\Big)^{\frac{1}{2}} \leqq \|\pi_{[y]} A\| \leqq \|A\|$$

for $A \in \mathscr{C}_c(G)$. Therefore any element A of $C^*(G)$ also corresponds to a continuous function on G, tending to 0 at infinity. In particular, (2.2) makes sense and defines a state.

The state $\hat{\alpha}$ is obviously (σ^t) invariant. If $A, B \in \mathscr{C}_c(G)$, then

$$\hat{\alpha}(\sigma_t A * B) = \int \alpha(dx) \sum_{y \in [x]} e^{iV(x,y)t} A(x, y) \cdot B(y, x)$$

extends to an entire function F of t. Using a partition of unity on supp A, we may write $A = \sum A_j$, where supp $A_j \in \mathcal{O}_j$, and $(\mathcal{O}_j, \varphi_j)$ is a conjugating homeomorphism, thus

$$F(t) = \sum_j \int \alpha(dx)\, A_j(x, \varphi_j x)\, B(\varphi_j x, x) \exp i\, V(x, \varphi_j x)\, t$$

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[3] For a discussion of the Kubo-Martin-Schwinger or KMS boundary condition, see for instance [25], [4]

[4] Note that Araki has discussed the relation between Gibbs states and KMS states in the more restricted case of lattice spin systems

and therefore

$$F(t+i)=\sum_j \int [e^{-V(x,\varphi_j x)}\alpha(dx)]\, B(\varphi_j x, x)\, A_j(x, \varphi_j x) \exp \mathrm{i}\, V(x, \varphi_j x)\, t.$$

If α is α Gibbs state we have thus

$$\begin{aligned} F(t+i) &= \sum_j \int \alpha(dy)\, B(y, \varphi_j^{-1} y)\, A_j(\varphi_j^{-1} y, y) \exp \mathrm{i}\, V(\varphi_j^{-1} y, y)\, t \\ &= \hat{\alpha}(B * \sigma_t A) \end{aligned}$$

so that $\hat{\alpha}$ satisfies the KMS condition. The converse is proved similarly.

2.2 Remarks. (i) *If α_0 is a Gibbs state corresponding to $V=0$, the corresponding state $\hat{\alpha}_0$ on $C^*(G)$ is a trace.*

(ii) For the purposes of Theorem 2.1, it suffices to consider $\hat{\alpha}$ restricted for $C_c(G)$, and the KMS condition thus restricted.

3. Smale spaces

Let Ω be a nonempty compact space, with a metric d, and a homeomorphism f: $\Omega \mapsto \Omega$. Following [15] we say that (Ω, d, f) is a Smale space if Ω has local product structure – with an "expanding" and a "contracting" direction – and if f (resp. f^{-1}) is a contraction for distances in the contracting (resp. expanding) direction. More explicitly we assume that $\varepsilon > 0$, $[\cdot,\cdot]$ and $\lambda \in (0, 1)$ exist such that the conditions (SS 1) and (SS 2) below are satisfied.

(SS 1) *The map*

$$\{(x, y) \in \Omega \times \Omega : d(x, y) < \varepsilon\} \mapsto [x, y] \in \Omega$$

is continuous; it satisfies $[x, x] = x$ and

$$[[x, y], z] = [x, z], \qquad [x, [y, z]] = [x, z]$$

when the two sides of these relations are defined.

Define

$$\begin{aligned} V_x^s(\delta) &= \{z : [x, z] = z \quad \textit{and} \quad d(x, z) < \delta\} \\ V_x^u(\delta) &= \{z : [z, x] = z \quad \textit{and} \quad d(x, z) < \delta\} \end{aligned}$$

One verifies that, for sufficiently small $d(x, y)$,

$$V_x^s(\varepsilon) \cap V_y^u(\varepsilon) = \{[x, y]\}.$$

Furthermore, $[\cdot,\cdot]$: $V_x^u(\delta) \times V_x^s(\delta) \mapsto \Omega$ is a homeomorphism onto an open subset of Ω for suitably small δ.

(SS 2) *The homeomorphism f satifies $f[x, y] = [fx, fy]$ when both sides are defined, and*

$$\begin{aligned} d(fy, fz) &\leqq \lambda\, d(y, z) \qquad \textit{if } y, z \in V_x^s(\delta) \\ d(f^{-1} y, f^{-1} z) &\leqq \lambda\, d(y, z) \qquad \textit{if } y, z \in V_x^u(\delta) \end{aligned}$$

If δ is sufficiently small, we have

$$V_x^s(\delta)=\{y: d(f^n x, f^n y)\leqq\delta \quad \text{for all } n\geqq 0\}$$
$$V_x^u(\delta)=\{y: d(f^{-n}x, f^{-n}y)<\delta \quad \text{for all } n>0\}$$

For more details, see [15].

Fried [7] has shown that d could be replaced by a Hölder equivalent metric such that besides (SS2) also the following properties are satisfied for suitable $\delta>0, L>0$

(SS3) *If* $d(x,y)<\delta$, *then* $d(x,[x,y])\leqq Ld(x,y)$

(SS4) f *and* f^{-1} *are Lipschitz.*

The nonwandering set Ω for a diffeomorphism f satisfying the Axiom A of Smale [24] has local product structure, and one can thus choose $[\cdot,\cdot]$ and a Riemann metric d such that all the above properties of a Smale space are satisfied. In fact it suffices (in view of Smale's *spectral decomposition*) to discuss the case where f is topologically transitive or even mixing on Ω. This gives the possibility of presenting an important part of the theory of hyperbolic dynamical systems in the abstract setting of Smale spaces.

The properties postulated in Section 2 are satisfied in the case of a Smale space, with $\Gamma=\mathbb{Z}$. Indeed f is expansive (see [15], Section 7.3) and satisfies condition (C) (see [15]), Section 7.15). In fact, if (x,y) is a conjugate pair, one obtains a conjugating homeomorphism $(\mathcal{O},\varphi)$ by writing

$$\varphi z=[f^{-n}[f^n[z,x],f^n y], f^n[f^{-n}y, f^{-n}[x,z]]]$$

when z is in a small open set $\mathcal{O}\ni x$, and n is suitably large. We shall associate Gibbs states with elements of $\mathscr{C}^\alpha(\Omega)$, the space of real Hölder continuous functions of exponent α on Ω. If $U\in\mathscr{C}^\alpha(\Omega)$, we define $V\colon G\mapsto\mathbb{R}$ by

$$V(x,y)=\sum_{k=-\infty}^{\infty}[U(f^k x)-U(f^k y)].$$

Since $d(f^k x, f^k y)\to 0$ exponentially fast, uniformly on compacts of G when $|k|\to\infty$. V is continuous. From there the definition of Gibbs states proceeds as in Section 2.

3.1. Theorem. *If f is topologically mixing on Ω, and $U\in\mathscr{C}^\alpha(\Omega)$, there is a unique Gibbs state ρ associated with A. This probability measure is τ-invariant, and is the unique τ-invariant probability measure making maximum the function*

$$\alpha\mapsto h_\tau(\alpha)+\alpha(U) \tag{3.1}$$

where h_τ is the Kolmogorov-Sinai invariant (entropy).

The study of Gibbs states on Smale spaces began with Sinai [23], and the use of Markov partitions. The Gibbs states defined with Markov partitions satisfy the variational principle (3.1) (see [15]) and are therefore independent of the choice of the partition. There remains however the problem of the equivalence with the definition given in Section 2. Half of the proof is easy (see [15]), the harder half was proved by Haydn [8].

3.2. *Remarks*. The formalism described above has two natural extensions.

(i) Replace continuous functions by β-Hölder continuous functions, and measures by *distributions* in $(\mathscr{C}^{\beta})^*$. This leads to a natural concept of *Gibbs distributions* on a Smale space.

(ii) Replace conjugating homeomorphisms by maps between stable manifolds along the unstable manifolds, and replace $V(x, y)$ by

$$V^s(x, y) = \sum_{k=-\infty}^{-1} [U(f^k x) - U(f^k y)].$$

One defines in this manner s-Gibbs states (or s-Gibbs distributions) which are measures (or distributions) on stable manifolds depending Hölder continuously on that manifold. (See [20], [16] for similar definitions in the framework of foliations). The u-Gibbs distributions are similarly defined, and the product of an s-Gibbs distribution by a u-Gibbs distribution is a Gibbs distribution in the sense of (i).

Definitions analogous to the above have in fact been made in the framework of symbolic dynamics (see [19]) and have proved useful in the discussion of *resonances* for hyperbolic systems. The above direct definitions are more natural because they do not use a Markov partition. Unfortunately the equivalence of the direct and "symbolic" definitions does not seem easy to establish, and the study of resonances requires at this time the "symbolic" definitions.

4. von Neumann algebras associated with invariant measures which have no zero characteristic exponent

In the last few years, the field of hyperbolic dynamics has been considerably widened, thanks to a current of ideas initiated by Pesin [12], [13] [5]. Roughly speaking, Pesin has shown that one can replace statements true uniformly by statements true almost everywhere with respect to an invariant measure ρ. In particular, hyperbolicity of a diffeomorphisms f is replaced by the condition that ρ has no zero characteristic exponent. Along these lines we shall show how the construction of a C^*-algebra in Sections 2, 3 can be replaced more generally by the construction of a von Neumann algebra together with a normal state $\hat{\rho}$ associated with ρ; *modular* groups of automorphisms will also be introduced.

It will be convenient to use the concept of rectangle defined as follows. If (V^s_ξ), (V^u_η) are families of local stable and unstable manifolds parametrized by ξ and η respectively, and if for each ξ, η the manifolds V^s_ξ, V^u_η have a single point of intersection $[\xi, \eta]$, which is furthermore transversal, then we say that

[5] See also Katok [9], Ledrappier and Young [10], etc

the set $R=\{[\xi,\eta]\}$ of all these intersections is a *rectangle* if R is compact. [This definition is inspired by that of Sinai and Bowen [1] for Axiom A basic sets, but we do not require here that R have dense interior].

4.1. Proposition. *Let M be a smooth compact manifold, and $f\colon M \mapsto M$ a diffeomorphism of class $C^{1+\alpha}$ (with $\alpha>0$).*

There is a set H with the following properties

(a) *H is an f-invariant Borel subset of M.*

(b) *$\rho(H)=1$ for every ergodic probability measure ρ with no zero characteristic exponent.*[6]

(c) *If $x \in H$, the stable and unstable manifolds of x are well defined and intersect transversally at x.*

Furthermore, if we introduce the (f,f) invariant set

$$G_H = \{(x,y) \in H \times H : y \text{ is on the intersection of the stable and unstable manifolds of } x\}.$$

we also have

(d) *G_H is the graph of an equivalence relation on H.*

(e) *Having chosen an invariant probability measure ρ one can replace H by a smaller set with again $\rho(H)=1$ such that G_H is a countable union of graphs of homeomorphisms $\varphi\colon R' \mapsto R''$ where R', R'' are rectangles*[7]*, and*

$$\lim_{|k|\to\infty} d(f^k x, f^k \varphi x)=0$$

uniformly for $x \in R'$.

This proposition results from the construction of the stable and unstable manifolds. Specifically, the constructions given in [12] and [18] yield all the properties listed.

From now on, Λ will be an f-invariant subset of H, which is a countable union of rectangles, and we define

$$G=G_\Lambda=G_H \cap (\Lambda \times \Lambda).$$

Therefore $(x,y)\in G$ is an equivalence relation on Λ. In the case of an Axiom A diffeomorphism one can take $\Lambda=$basic set.

4.2. Remark. An equivalence class for the relation $(x,y)\in G$ is the set of all transversal intersection points in Λ of a given stable and a given unstable manifolds. Because of the transversality of these intersections, *the equivalence class* $[x]$ *of each $x\in\Lambda$ is countable.* In particular, for $\Lambda=H$, we see why G_H can be taken as a *countable* union of graphs of maps $\varphi\colon R' \mapsto R''$. Similarly, it follows from Proposition 4.1 that G is a countable union of graphs of homeomorphisms $\varphi_i\colon \Lambda_i' \mapsto \Lambda_i''$ where Λ_i', Λ_i'' are rectangles.

[6] Note that (b) is equivalent to the requirement that $\rho(H)=1$ for every f-invariant probability measure ρ such that the characteristic exponents are ρ-almost everywhere nonzero

[7] Note that in Sections 2, 3 we used open sets instead of rectangles, but that an open set in a Smale space is a countable union of rectangles

4.3. Definition. *Let V: $G \mapsto \mathbb{R}$ satisfy $V(x,y)+V(y,z)=V(x,z)$. We say that the probability measure ρ on Λ is a Gibbs state with respect to V if, whenever φ is a homeomorphism of a rectangle $\Lambda' \subset \Lambda$ to $\Lambda'' \subset \Lambda$, with graph contained in G, the image $\varphi(\rho/\Lambda')$ is absolutely continuous with respect to $\rho|\Lambda''$ and has Radon-Nicodym derivative*

$$y \mapsto h(y) = \exp V(\varphi^{-1} y, y).$$

We assume that the obvious measurability requirement on V is satisfied. Note that it suffices to verify the above condition for the countable family (φ_i) of Remark 4.2. Our definition does not require that V and ρ be invariant (under (f,f) and f respectively), but the V's which we shall consider are of the form

$$V(x,y) = \sum_{k=-\infty}^{\infty} [U(f^k x) - U(f^k y)]$$

and therefore invariant.

For any probability measure ρ on Λ we can define the complex Hilbert space $\mathfrak{H} = \mathfrak{H}_\rho$ of ρ-square integrable functions Ψ: $y \mapsto l^2([y])$ on Λ, where $l^2([y])$ is the space of square summable functions $[y] \mapsto \mathbb{C}$. An element Ψ of $\mathfrak{H}$ is thus a function $G \mapsto \mathbb{C}$ such that

$$\|\Psi\|^2 = \int \rho(dy) \sum_{x \in [y]} |\Psi(x,y)|^2 < \infty.$$

Given a function A: $G \mapsto \mathbb{C}$, we write

$$\|A\|_{[x]} = \sup_{\xi} \frac{\|\sum_{z \in [x]} A(\cdot, z)\, \xi(z)\|_{l^2([x])}}{\|\xi\|_{l^2([x])}}$$

and

$$\|A\| = \operatorname{ess.\,sup.}_x \|A\|_{[x]}$$

where the essential sup is with respect to $\rho(dx)$. Those A for which $\|A\| < \infty$ define operators on $\mathfrak{H}$ by the formula

$$(A\Psi)(x,y) = \sum_{z \in [u]} A(x,z)\, \Psi(z,y).$$

It is easily seen that the set of these operators is a von Neumann algebra $\mathscr{B}$, and that $\|A\|$ is the operator norm.

4.4. Remark. The center of $\mathscr{B}$ consists of those C: $G \mapsto \mathbb{C}$ such that $C(x,y)=0$ if $x \neq y$ and $C(x,x)$ depends only on $[x]$. In particular, if a ρ-measurable function which is constant on equivalence classes is almost everywhere constant, then $\mathscr{B}$ is a factor.

If $V: G \mapsto \mathbb{R}$ satisfies $V(x, y)+V(y, z)=V(x, z)$, a one parameter group (σ^t) of $*$-automorphisms of $\mathscr{B}$ is defined by

$$(\sigma^t A)(x, y)=e^{iV(x,y)t} A(x, y) \tag{4.1}$$

4.5. Theorem. (i) *For every probability measure ρ on Λ, there is a vector state $\hat{\rho}$ on $\mathscr{B}_\rho$ such that*

$$\hat{\rho}(A)=\int_\Lambda \rho(dx)\, A(x, x)$$

(ii) *$\hat{\rho}$ satisfies the* KMS *condition with respect to (σ^t) if and only if it is a Gibbs state with respect to V.*

Concerning (i), let $\Phi \in \mathfrak{H}$ be such that $\Phi(x, y)=\delta_{xy}$, then $\|\Phi\|=1$ and

$$(\Phi, A\Phi)=\int_\Lambda \rho(dx)\, A(x, x)=\hat{\rho}(A)$$

so that $\hat{\rho}$ is a vector state (hence normal).

Part (ii) is an easy extension of the proof of part (ii) of Theorem 2.1.

4.6. Remark. If ρ_0 is a Gibbs state corresponding to $V=0$, the corresponding state $\hat{\rho}_0$ on $\mathscr{B}$ is a trace. In particular, if $\mathscr{B}$ is an infinite dimensional factor, it is of type II_1.

4.7. Problem. Let $U: M \mapsto \mathbb{R}$ be Hölder continuous, and

$$V(x, y)=\sum_{k=-\infty}^{\infty} [U(f^k x)-U(f^k y)] \tag{4.2}$$

what is the relation between Gibbs states with respect to V and f-invariant probability measures ρ making $h(\rho)+\rho(U)$ maximum? If f is C^∞, it is known that $\rho \mapsto h(\rho)$ is upper semi-continuous (Newhouse [11]) and therefore, if U is continuous[8], $h(\rho)+\varrho(U)$ reaches its maximum on a nonempty set of *equilibrium states.* (This is a Choquet simplex, and its extremal points are ergodic). It is then natural to conjecture that such equilibrium states are Gibbs states with respect to V defined by (4.2), for suitable Λ.

4.8. States with local product structure

Let us consider the special case $U=0$. We assume therefore that ρ makes the entropy $h(\rho)$ maximum (and we may take ρ ergodic, as discussed above). In the Axiom A case (see Bowen [1]), we know that ρ has local product structure. Conjecture: in general, if ρ makes the entropy maximum, and has no zero characteristic exponents, then it has local product structure. Let us give a precise definition. Suppose that a rectangle R consists of the intersections of local stable and unstable manifolds of the families (V_ξ^s), (V_z^u) parametrized by ξ and z respec-

[8] Hölder continuity is not used here, but it is needed to prove the convergence of (4.2)

tively. If, for every such rectangle R, the restriction $\rho|R$ is of the form $\rho^s(d\xi) \times \rho^u(dz)$, then we say that ρ has *local product structure.*

Suppose now that ρ is an f-invariant probability measure with local product structure. Let φ be a homeomorphism of a rectangle $\Lambda' \subset M$ to a rectangle $\Lambda'' \subset M$, such that the graph of φ is contained in G and

$$\lim_{|k| \to \infty} d(f^k x, f^k \varphi x) = 0$$

uniformly in Λ'. Then, using a covering of $f^k(\Lambda' \cup \Lambda'')$ by rectangles, for large positive or negative k, we see that $\varphi(\rho|\Lambda')$ is proportional to $\rho|\Lambda''$ with locally constant proportionality factor. More precisely, we can write

$$a\varphi(\rho|\Lambda') + b(\rho|\Lambda'') = 0.$$

Thus, states with maximum entropy, states with local product structure, and Gibbs states with $V=0$ are all closely related, but they are known to coincide only for mixing Axiom A basic sets.

4.9. SRB states

If f: $M \mapsto M$ is a C^2 diffeomorphism of a compact manifold, Ledrappier and Young [10] have shown that the following two conditions on an f-ergodic measure ρ are equivalent.

(a) The entropy $h(\rho)$ is equal to the sum of the positive characteristic (Lyapunov) exponents of ρ (taking into account multiplicity).

(b) The conditional measures of ρ on unstable manifolds are absolutely continuous with respect to Lebesgue (i.e. smooth) measure on these unstable manifolds.

If these conditions are satisfied, ρ is called an SRB measure [9] and it is shown in [10] that the densities of conditional measures in (b) are actually C^1.

Suppose now that ρ is an SRB measure, and has no zero characteristic exponents. Let φ be a homeomorphism of a rectangle Λ' to a rectangle Λ'' such that the graph of φ is in G and $\lim_{|k| \to \infty} d(f^k x, f^k \varphi x) = 0$ uniformly. We can label a point of Λ' by its coordinates ξ, η in the unstable and stable directions respectively, and the corresponding point $\varphi([\xi, \eta])$ of Λ'' by its coordinates $\psi(\xi, \eta), \eta$ where it is permissible and convenient to use the same coordinate η in the stable direction. The measure $\rho|\Lambda'$ is then of the form $a_1(\xi, \eta)\, d\sigma \times \rho_1(d\eta)$ where $d\sigma$ denotes the Riemann volume element in an unstable manifold after choice of some Riemann metric on M; the density $a_1(\xi, \eta)$ is a C^1 function

[9] These measures have been investigated in the Axiom A case by Sinai [23], Ruelle [14], Bowen and Ruelle [3]. The equivalence of (a) and (b) was conjectured in [17], but the general proof (which is not easy) was only obtained in [10]

of η, in Λ'. Similarly, the measure $\rho|\Lambda''$ is of the form $a_2(\xi,\eta)\, \mathrm{d}\sigma \times \rho_1(\mathrm{d}\eta)$ with the same $\rho_1(d\eta)$. Finally, $\varphi(\rho|\Lambda')$ is of the form $a_3(\xi,\eta)\, d\sigma \times \rho_1(d\eta)$. We have

$$\frac{a_2(\xi',\eta)}{a_1(\xi,\eta)} = \prod_{k=-\infty}^{-1} \frac{J^u(f^k[\xi,\eta])}{J^u(f^k[\xi',\eta])}$$

where J^u is the "Jacobian determinant" of f in the unstable direction (defined using the Riemann metric). On the other hand

$$\frac{a_3(\psi(\xi,\eta),\eta)}{a_1(\xi,\eta)} = \prod_{k=0}^{\infty} \frac{J^u(f^k(\psi(\xi,\eta),\eta))}{J^u(f^k(\xi,\eta))}$$

so that

$$\frac{a^3(\psi(\xi,\eta),\eta)}{a_2(\psi(\xi,\eta),\eta)} = \exp V([\xi,\eta],\varphi[\xi,\eta])$$

where we have written

$$V(x,y) = \sum_{k=-\infty}^{\infty} [\log J^u(f^k y) - \log J^u(f^k x)]. \tag{4.3}$$

Therefore $\varphi(\rho|\Lambda')$ is absolutely continuous with respect to $\rho|\Lambda''$, and has Radon-Nicodym derivative

$$y \mapsto \exp V(\varphi^{-1} y, y)$$

so that ρ is a Gibbs state with respect to V defined by (4.3). We have proved the following.

4.10. Theorem. *Let f: $M \mapsto M$ be a C^2 diffeomorphism of a compact manifold, and ρ an f-ergodic measure. If ρ is* SRB *and has no zero characteristic exponent, then ρ is a Gibbs state with respect to V defined by* (4.3). *The corresponding state $\hat{\rho}$ is therefore* KMS *with modular group defined by* (4.1).

References

1. Bowen, R.: Markov partitions for Axiom A diffeomorphisms. Am. J. Math. **92**, 725–747 (1970)
2. Bowen, R.: Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Math. vol. 470. Berlin-Heidelberg-New York: Springer 1975
3. Bowen, R., Ruelle, D.: The ergodic theory of Axiom A flows. Invent. Math. **29**, 181–202 (1975)
4. Bratteli, O., Robinson, D.W.: Operator algebras and quantum statistical mechanics. Berlin-Heidelberg-New York: Springer (I) 1979 (II) 1981
5. Capocaccia, D.: A definition of Gibbs state for a compact set with Z^ν action. Commun. Math. Phys. **48**, 85–88 (1976)
6. Connes, A.: A survey of foliations and operator algebras. Proc. Symp. Pure Math. **38**(1), 521–628 (1982)
7. Fried, D.: Métriques naturelles sur les espaces de Smale. C.R. Acad. Sc. Paris, Sér. I, **297**, 77–79 (1983)
8. Haydn, N.T.A.: On Gibbs and equilibrium states. Ergodic. Theory. Dyn. Syst. **7**, 119–132 (1987)

9. Katok, A.: Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. Publ. Math., Inst. Hautes Etud. Sci. **51**, 137–174 (1980)
10. Ledrappier, F., Young, L.-S.: The metric entropy of diffeomorphisms Part I. Characterization of measures satisfying Pesin's formula. Part II. Relations between entropy, exponents and dimension. Ann. Math. **122**, 509–539, 540–574 (1985)
11. Newhouse, S.: Continuity properties of entropy. Preprint
12. Pesin, Ya.B.: Invariant manifold families which correspond to nonvanishing characteristic exponents. Izv. Akad. Nauk SSSR, Ser. Mat. **40** (No. 6) 1332–1379 (1976) (English translation), Math. USSR, Izv. **10**, (No. 6) 1261–1305 (1976)
13. Pesin, Ya.B.: Lyapunov characteristic exponents and smooth ergodic theory. Usp. Mat. Nauk **32**, (No. 4); 55–112 (1977) (English translation). Russ. Math. Surv. **32**, (No. 4) 55–114 (1977)
14. Ruelle, D.: A measure associated with Axiom A attractors. Am. J. Math. **98**, 619–654 (1976)
15. Ruelle, D.: Thermodynamic formalism. Encyclopedia of Math. and its Appl., Vol. 5. Mass: Addison-Wesley, Reading 1978
16. Ruelle, D.: Integral representation of measures associated with a foliation. Publ. Math., Inst. Hautes Etud, Sci. **48**, 127–135 (1978)
17. Ruelle, D.: Sensitive dependence on initial condition and turbulent behavior of dynamical systems. Ann. N.Y. Acad. Sci. **316**, 408–416 (1979)
18. Ruelle, D.: Ergodic theory of differentiable dynamical systems. Pub. Math., Inst. Hautes Etud. Sci. **50**, 275–306 (1979)
19. Ruelle, D.: One-dimensional Gibbs states and Axiom A diffeomorphisms. J. Differ. Geom. **25**, 117–137 (1987)
20. Ruelle, D., Sullivan, D.: Currents, flows and diffeomorphisms. Topology **14**, 319–327 (1975)
21. Sinai, Ia.G.: Markov partitions and C-diffeomorphisms. Funkts. Anal. Prilozh. **2**, (No. 1) 64–89 (1968); (English translation) Funct. Anal. Appl. **2**, 61–82 (1968)
22. Sinai, Ia.G.: Construction of Markov partitions. Funkts. Anal. Prilozh. **2**, (No. 3) 70–80 (1968); (English translation). Funct. Anal. Appl. **2**, 245–253 (1968)
23. Sinai, Ia.G.: Gibbs measures in ergodic theory. Usp. Mat. Nauk **27** (No. 4) 21–64 (1972); (English translation) Russ. Math. Surv. **27**, (No. 4) 21–69 (1972)
24. Smale, S.: Differentiable dynamical systems. Bull. Am. Math. Soc. **73**, 747–817 (1967)
25. Takesaki, M.: Tomita's theory of modular Hilbert algebras and its applications. (Lect. Notes Math., vol. 128). Berlin-Heidelberg-New York: Springer 1970
26. Winkelnkemper, E.: The graph of a foliation. Preprint

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Note added in proof

For a result related to theorem 4.10 we refer the reader to F. Ledrappier, Propriétés ergodic des mesures de Sinai. C.R. Acad. Sci. Paris **294**, 593–595 (1982)