n-TOADS SEP 15TH

NOTES BY ANTHONY, SLIGHTLY EDITED BY MARCELO

1. Structure/Plan

- C^*-algebras - initially, think of $M_n(\mathbb{C})$, the complex, $n \times n$ matrices;
- KMS States: "Equilibrium states of quantum dynamical systems"
- Construction/analysis of KMS states for matrix C^*-algebras (Anna).
- Cuntz-Krieger and graph C^*-algebras (Marcelo);
- C^*-algebras from Number Theory (Bost-Connes systems, C^*-algebras of affine monoids of algebraic integers (CBruce?);
- Open dynamical systems (CBose);
- Equilibrium states of dynamical systems;
- Gibbs states / SRB measures (Anthony?);
- Smale spaces (Ian);
- Connections with Perron-Frobenius theory, class field theory, and with Furstenberg's conjecture in ergodic theory...

- Principle: *Where KMS States lead you is interesting Mathematics.*

2. Hints of what it's all about

Underlying data (e.g. directed graph, number field, Smale space, billiard with holes, etc...) gives rise to (A, σ).

C^*-dynamical system (A, σ). A is a *non-commutative C^*-algebra*; σ time evolution or dynamics. Model for quantum evolution in the Heisenberg picture: the state of the system remains fixed, while the observables (i.e. self adjoint elements of A) change with time under a continuous one-parameter automorphism group σ.

State: positive linear functional of norm 1. (this is the quantum (or noncommutative) analog of the integral with respect to a probability distribution on the state space of a classical system).

Definition: A *trace* on a C^*-algebra A is a state of A satisfying $\tau(ab) = \tau(ba)$. The KMS condition is a generalization of this when one has a time evolution σ: φ is a KMS state of A if

$$\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$$

for appropriately chosen $a, b \in A$.

KMS states (states of equilibrium) form Choquet simplex K_β. Here β is a real parameter. By the Choquet simplex property, every state in K_β is a unique (generalized) convex linear combination of extremal points of K_β.

We denote by E_β the set of extremal KMS$_\beta$ states. Extremal points are factor states, namely, their associated (GNS) representations have trivial center, and are thus considered "pure phases".

Fact K_β is a dynamical invariant of the system (A, σ).

Some evidence, to be pursued further:

Ruelle: Smale spaces have C^*-algebras and dynamics for which K_β corresponds to Gibbs measures; "extremal" corresponds to "ergodic".

C*-algebra of a directed graph (or for a Cuntz-Krieger algebra): there is (a choice of) dynamics such that the KMS states encode information about the graph; at least the Perron Frobenius eigenvalue and eigenvector for the various connected components of the connectivity matrix of the graph.

Number fields: the KMS states of associated C^*-algebras are useful to understand the number field. (Dedekind zeta function, class number, etc.)

Open dynamical systems: *we believe* there should be an associated C*-algebra (possibly generated by a Hilbert bimodule or correspondence over a commutative algebra?) for which the dynamics has KMS states associated to the quasi-invariant measures of the open system.

In all these constructions there is a somewhat specialized setup: $A =$ generalized cross product C^*-algebra; $\sigma =$ subgroup of a dual action or co-action. But things seem flexible enough to include lots of new situations.

Main Idea:

want to look for invariants of the original data coming from the equilibrium states. Ergodic theorists (allegedly) have refined tools to describe ergodic invariant measures, and are interested in various aspects of these measures. Want to import these ideas into KMS state picture. But also KMS states have stability properties (beyond just invariance), and these would have implications for the ergodic measures.

Warning: classical introductions (i.e. pre-1970) to KMS states of C^*-algebras tend to be oriented heavily towards physical systems, and this is too restrictive for many of the systems of interest to us.

Interesting special case to begin with: matrix C^*-algebras. With a bit of linear algebra (finite dimensional operator theory) we can compute everything explicitly here... next week.