KMS-STATES

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CONTENTS

ABSTRACT

In the first section, we will repeat basic definitions regarding C^*-algebras and gather complex analytic tools that will be used. We will then follow [8], Section 8.12, to introduce α-KMS-states in the second section. The equivalence of our and the original definition is shown in Theorem 2.3, including a slight improvement in generality in comparison to [8], and we will see that all α-KMS-states are α-invariant. Lastly, we will go through some details of the matrix example as it was done in [6], Section 1.3.

1 PREREQUISITES

1.1 Basic C^*-algebraic definitions

For more information about the following definitions and constructions, the reader is referred to [3], Section II.1.

A *Banach space* is a normed complex vector space which is complete. For a linear map $T\colon X \to Y$ between two Banach spaces, we define

$$\|T\|_{\text{op}} := \sup_{\|x\| \le 1} \|Tx\|.$$

A *C^*-algebra* A is a Banach space which additionally is an algebra and is equipped with a map $^*\colon A \to A$ such that the following conditions hold for every $a, b \in A$ and $\lambda \in \mathbb{C}$:

- $(a+b)^* = a^* + b^*$ and $(\lambda a)^* = \overline{\lambda} a^*$ (* is *conjugate-linear*)
- $(a \cdot b)^* = b^* \cdot a^*$
- $(a^*)^* = a$
- $\|a \cdot b\| \le \|a\| \cdot \|b\|$ (*submultiplicativity*)
- $\|a^*\| = \|a\|$ (* is *isometric*)
- $\|a^* \cdot a\| = \|a\|^2$ (*C^*-condition*)

A *homomorphism* between two C^*-algebras A, B is a linear map $\phi\colon A \to B$ which is multiplicative and *-preserving, and we will denote by $\mathsf{Aut}(A)$ the group of automorphisms of A. A tuple (A, α) consisting of a C^*-algebra A and a strongly continuous[1] group homomorphism $\alpha\colon \mathbb{R} \to \mathsf{Aut}(A)$ will be called a *C^*-dynamical system.*

A bounded linear functional $\varphi\colon A \to \mathbb{C}$ is *positive* (in symbols, $\varphi \ge 0$) if $\varphi(a^*a) \ge 0$ for all $a \in A$. Note that if A is unital, then the norm of such φ is given by $\|\varphi\|_{\text{op}} = \varphi(1_A)$ (see [3], II.6.2.5). We call a positive linear functional *state* if $\|\varphi\|_{\text{op}} = 1$. A state φ is called *pure* if, whenever there are states ψ_1, ψ_2 and $\lambda \in (0,1)$ such that $\varphi = \lambda\psi_1 + (1-\lambda)\psi_2$, then $\psi_1 \equiv \psi_2 \equiv \varphi$.

1.2 Complex Analysis Weaponry

We will write $\mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) \ge 0\}$ and $\mathring{\mathbb{H}}$ for its interior in $\mathbb{C}$.

Lemma 1.2.1 (cf. [9], Theorem 10.28)**.** *If $(f_n)_n$ is a sequence of functions which are analytic on a region Ω and which converge uniformly on compact subsets to a function f, then f is analytic.*

Lemma 1.2.2. *If f is an analytic function on the region Ω, then the map $\tilde{f}(z) := \overline{f(\overline{z})}$ is analytic on $\overline{\Omega} := \{z : \overline{z} \in \Omega\}$.* □

1 That is, for every $a \in A$, the map $G \to A, s \mapsto \alpha_s(a)$ is continuous.

Proposition 1.2.3 (Phragmén–Lindelöf Theorem; cf. [9], Theorem 12.8)**.** *Let* $a, b \in \mathbb{R}$ *and* $\Omega := \{s + \mathbf{i}t : a \le t \le b\}$. *Suppose we are given* $g \in \mathrm{C_b}(\Omega)$ *which is analytic on* $\mathring{\Omega}$. *Then for any* $a \le t \le b$, *we have*

$$\sup_{s\in\mathbb{R}} |g(s+\mathbf{i}t)| \le \sup_{s\in\mathbb{R}} |g(s+\mathbf{i}a)|^{b-t} \cdot \sup_{s\in\mathbb{R}} |g(s+\mathbf{i}b)|^{t}.$$

Proposition 1.2.4 (Generalized Phragmén–Lindelöf Theorem; cf. [11], Theorem 5)**.** *Suppose* f *is analytic on* $\mathring{\mathbb{H}}$ *and continuous on its boundary such that* $|f(t)| < 1$ *for all* $t \in \mathbb{R}$. *Suppose further that there exists a Lebesgue-integrable function* $\psi : (0, \pi) \to \mathbb{R}$ *so that for every* $\delta > 0$, *there is a number* $R(\delta)$ *such that*

$$\left|f\left(r\mathrm{e}^{\mathbf{i}\theta}\right)\right| \le \mathrm{e}^{r\delta \mathrm{e}^{\psi(\theta)}} \quad \textit{whenever } r > R(\delta), \theta \in (0, \pi).$$

Then $|f(z)| \le 1$ *for all* $z \in \mathbb{H}$.

Proposition 1.2.5 (Schwarz Reflection Principle; cf. [9], Theorem 11.14)**.** *Suppose* L *is a segment of the real axis,* Ω *is a region in* $\mathring{\mathbb{H}}$, *and every* $t \in L$ *is the centre of an open disc* D_t *such that* $\mathring{\mathbb{H}} \cap D_t \subset \Omega$. *Suppose* $f = u + \mathbf{i}v$ *is analytic on* Ω, *and*

$$\lim_{n\to\infty} v(z_n) = 0$$

for every sequence $(z_n)_n$ *in* Ω *which converges to a point of* L. *Then there is a function* F *which is analytic on* $\Omega \cup L \cup \overline{\Omega}$ *such that*

$$F(\overline{z}) = \overline{F(z)} \quad \textit{for all } z \in \Omega \cup L \cup \overline{\Omega}.$$

Corollary 1.2.6. *Suppose* f *is a function which is analytic on a strip* $\Omega := \{z : a < \mathrm{Im}(z) < b\}$. *Suppose further that* f *is continuous on the lower (resp. upper) boundary of* Ω, *and that there is an entire function* g *such that*

$$(g-f)|_{\mathbb{R}+\mathbf{i}a} \equiv 0 \quad (\textit{resp. } (g-f)|_{\mathbb{R}+\mathbf{i}b} \equiv 0).$$

Then f *and* g *coincide on* Ω.

Proof. By translation and Lemma 1.2.2, we can without loss of generality assume that $a = 0$ and that $(g-f)|_{\mathbb{R}} \equiv 0$. By the Schwarz Reflection Principle, the function

$$H(z) = \begin{cases} \big(g-f\big)(z) & \text{if } z \in \Omega \cup \mathbb{R} \\ \overline{\big(g-f\big)(\overline{z})} & \text{if } z \in \overline{\Omega} \cup \mathbb{R} \end{cases}$$

is analytic on the region $\Omega \cup \mathbb{R} \cup \overline{\Omega}$, which contains the real axis. Since $H(t) = 0$ for all $t \in \mathbb{R}$, the Identity Theorem yields $H \equiv 0$, and we are done. □

1.3 Analyticity

For this subsection, let X be a Banach space and let X^* be its space of continuous linear functionals $\varphi\colon X \to \mathbb{C}$.

Definition 1.3.1. A function $F\colon \mathbb{C} \to X$ is called (*weak**) *analytic* if for every $\varphi \in X^*$, the function $\varphi \circ F\colon \mathbb{C} \to \mathbb{C}$ is entire.

Lemma 1.3.2. *If $F, G\colon \mathbb{C} \to X$ are both analytic extensions of a map $f\colon \mathbb{R} \to X$, then $F \equiv G$.*

Proof. By assumption, the functions $\varphi \circ F$ and $\varphi \circ G$ are entire for each $\varphi \in X^*$. Since F and G are extensions of f, we know

$$(\varphi \circ F)|_{\mathbb{R}} = \varphi \circ f = (\varphi \circ G)|_{\mathbb{R}},$$

so by the Identity Theorem, $\varphi \circ F \equiv \varphi \circ G$ on all of $\mathbb{C}$. As φ was arbitrary, the Hahn–Banach Theorem implies $F \equiv G$. □

Lemma 1.3.3. *If $F\colon \mathbb{C} \to X$ is analytic, then for any bounded linear map $T\colon X \to Y$ into another Banach space Y, the function $T \circ F$ is also analytic. If $h\colon \mathbb{C} \to \mathbb{C}$ is an entire function, then $F \circ h$ is analytic.*

Proof. The first part follows immediately from the fact that $\varphi \circ T$ is an element of X^* for $\varphi \in Y^*$.

The second part follows from the fact that for any $\varphi \in X^*$, the function $\varphi \circ (F \circ h) = (\varphi \circ F) \circ h$ is entire, as the composition of entire functions. □

Remark 1.3.4 (cf. [8], Appendix 4)**.** It can be shown that a function $F\colon \mathbb{C} \to X$ is analytic if and only if for every $z_0 \in \mathbb{C}$, there is a neighbourhood U of z_0 and a sequence $(x_n)_n$ in X such that for all $z \in U$:

$$F(z) = \sum_{n=0}^{\infty} (z - z_0)^n x_n.$$

For the rest of this work, fix a C^*-dynamical system (A, α).

Definition 1.3.5. An element $x \in A$ is called (α-)*analytic* if the continuous function $\mathbb{R} \to A$, $t \mapsto \alpha_t(x)$, has an analytic extension $\mathbb{C} \to A$, $z \mapsto \alpha_z(x)$ (which is unique according to Lemma 1.3.2).

Lemma 1.3.6. *For an analytic element x of A, we have for all $z, w \in \mathbb{C}$ that $\alpha_{w+z}(x) = \alpha_w\big(\alpha_z(x)\big)$.*

Proof. Let us fix z. It is not clear a priori that $\alpha_z(x)$ is also an analytic element, that is, that the right-hand side of the asserted equation is defined at all. The left-hand side, however, makes sense. The map $w \mapsto \alpha_{w+z}(x)$ is the composition of the analytic map $F\colon w \mapsto \alpha_w(x)$ with the entire function $h_z\colon w \mapsto w + z$, and so it is analytic itself by Lemma 1.3.3. In order to show the claim, we have to explain that this map is an extension of the map $s \mapsto \alpha_s\big(\alpha_z(x)\big)$:

For $s \in \mathbb{R}$, the map $G_s\colon z \mapsto \alpha_s\big(\alpha_z(x)\big)$ is analytic by Lemma 1.3.3, and

$$\big(F \circ h_s\big)|_{\mathbb{R}} = G_s|_{\mathbb{R}} \quad \text{due to } \alpha_{t+s} = \alpha_{s+t} = \alpha_s \circ \alpha_t \text{ for } t \in \mathbb{R}.$$

Hence by Lemma 1.3.2, we have $F \circ h_s \equiv G_s$. In particular for our fixed $z \in \mathbb{C}$, we get

$$\alpha_{s+z}(x) = \alpha_{z+s}(x) = F \circ h_s(z) = G_s(z) = \alpha_s\big(\alpha_z(x)\big).$$

As s was arbitrary, we conclude that $w \mapsto \alpha_{w+z}(x)$ indeed extends $s \mapsto \alpha_s\big(\alpha_z(x)\big)$. □

Theorem 1.3.7. *The set A^a of analytic elements is dense in A.*

In order to prove this, we first need to recap vector- (that is, Banach space-) valued integrals, as seen in [7], end of Section 2.5, for example.

Proposition 1.3.8. *If $f\colon \mathbb{R} \to X$ is a continuous function with $\|f(\cdot)\|$ Lebesgue integrable, then there is a unique element $\int f \,\mathrm{d}\lambda$ of X such that*

$$\varphi\Big(\int f(t)\,\mathrm{d}\lambda\Big) = \int_{\mathbb{R}} \big(\varphi \circ f\big)(t)\,\mathrm{d}t \qquad \textit{for all } \varphi \in X^*. \tag{1}$$

Moreover:

a) Integration by substitution: *If $g\colon \mathbb{R} \to \mathbb{R}$ is a continuously differentiable bijective function, then*
$$\int f\big(g(t)\big)g'(t)\,\mathrm{d}\lambda = \int f(t)\,\mathrm{d}\lambda.$$

b) *If $T : X \to Y$ is a bounded linear map into another Banach space Y, then*
$$T\Big(\int f(t)\,\mathrm{d}\lambda\Big) = \int \big(T \circ f\big)(t)\,\mathrm{d}\lambda.$$

Proof. The existence and uniqueness of the element $\int f\,\mathrm{d}\lambda$ with Property (1) follows as a corollary of [7], Prop. 2.5.15.

ad a): This follows directly from the classical result which holds for the right-hand side of Equation (1).

ad b): Since $\|T \circ f(\cdot)\| \le \|T\|_{\mathrm{op}}\|f(\cdot)\|$ is integrable, the right-hand side is defined. Now the claim follows from the fact that $\psi \circ T$ is an element of X^* for each $\psi \in Y^*$. □

Proof of Theorem 1.3.7. For an arbitrary fixed $x \in A$, define

$$x_n := \sqrt{\frac{n}{\pi}} \int \alpha_t(x)\exp\left(-nt^2\right)\mathrm{d}\lambda \quad \in A$$

and

$$F_n : \mathbb{C} \to A, \quad z \mapsto \sqrt{\frac{n}{\pi}} \int \alpha_t(x) \exp\left(-n(t-z)^2\right) \mathrm{d}\lambda,$$

using Proposition 1.3.8. The function F_n is analytic (see for example [2], Theorem 25; a generalized version of Lebesgue's Dominated Convergence Theorem). For $s \in \mathbb{R}$, we get

$$\begin{aligned} F_n(s) &= \sqrt{\frac{n}{\pi}} \int \alpha_t(x) \exp\left(-n(t-s)^2\right) \mathrm{d}\lambda \\ &\overset{1.3.8a)}{=} \sqrt{\frac{n}{\pi}} \int \alpha_{s+t}(x) \exp\left(-nt^2\right) \mathrm{d}\lambda \\ &\overset{1.3.8b)}{=} \alpha_s\left(\sqrt{\frac{n}{\pi}} \int \alpha_t(x) \exp\left(-nt^2\right) \mathrm{d}\lambda\right), \end{aligned}$$

so that F_n is an extension of $s \mapsto \alpha_s(x_n)$. Hence, $x_n \in A^a$.

We now want to see that x_n converges to x in norm. Using Proposition 1.3.8a), we rewrite

$$x_n = \sqrt{\frac{n}{\pi}} \int \alpha_{\frac{1}{\sqrt{n}}t}(x) \exp\left(-t^2\right) \frac{1}{\sqrt{n}} \mathrm{d}\lambda = \sqrt{\frac{1}{\pi}} \int \alpha_{\frac{1}{\sqrt{n}}t}(x) \exp\left(-t^2\right) \mathrm{d}\lambda$$

$$\text{and} \quad x = \sqrt{\frac{1}{\pi}} \int x \exp\left(-t^2\right) \mathrm{d}\lambda$$

to get

$$\begin{aligned} \sqrt{\pi}\, \|x - x_n\| &= \left\| \int \left(x - \alpha_{\frac{1}{\sqrt{n}}t}(x)\right) \exp\left(-t^2\right) \mathrm{d}\lambda \right\| \\ &= \sup\left\{ \left| \int_{\mathbb{R}} \varphi\left(x - \alpha_{\frac{1}{\sqrt{n}}t}(x)\right) \exp\left(-t^2\right) \mathrm{d}t \right| : \varphi \in A^*, \|\varphi\| \le 1 \right\} \\ &\le \int_{\mathbb{R}} \left\| x - \alpha_{\frac{1}{\sqrt{n}}t}(x) \right\| \exp\left(-t^2\right) \mathrm{d}t. \end{aligned}$$

Since α is strongly continuous, the integrand of the last line converges pointwise to zero. Since it is also dominated by the integrable function $t \mapsto 2\|x\| \exp\left(-t^2\right)$, Lebesgue's Dominated Convergence Theorem yields

$$\|x - x_n\| \to 0.$$

□

Proposition 1.3.9. *A^a is a *-subalgebra of A.*

Proof. It is obvious that the linear combination of analytic elements is again analytic. For self-adjoint $\varphi \in A^*$ and $a \in A$, we have $\varphi(a^*) = \overline{\varphi(a)}$, so if $x \in A^a$, the function

$$z \mapsto \varphi\left(\alpha_{\overline{z}}(x)^*\right) = \overline{\varphi\left(\alpha_{\overline{z}}(x)\right)}$$

is entire by Lemma 1.2.2. By [3], Theorem II.6.3.4, any $\psi \in A^*$ can be written as $\psi = \varphi_1 + \mathbf{i}\varphi_2$ where φ_i is self-adjoint, so that the function

$$z \mapsto \psi\left(\alpha_{\overline{z}}(x)^*\right) = \varphi_1\left(\alpha_{\overline{z}}(x)^*\right) + \mathbf{i}\varphi_2\left(\alpha_{\overline{z}}(x)^*\right)$$

is a linear combination of entire functions and thus itself entire. Hence, $F\colon z \mapsto \alpha_{\overline{z}}(x)^*$ is analytic. Since we have $\alpha_s(x^*) = \alpha_s(x)^*$ for $s \in \mathbb{R}$, we infer that F extends $s \mapsto \alpha_s(x^*)$, so that $x^* \in A^a$.

For y another analytic element of A, the map $z \mapsto \alpha_z(x)\alpha_z(y)$ is analytic, for example by Remark 1.3.4. Since α_s is a homomorphism for $s \in \mathbb{R}$, we have $\alpha_s(x)\alpha_s(y) = \alpha_s(xy)$ and so we have found the analytic extension of $s \mapsto \alpha_s(xy)$. □

2 THE KMS CONDITION

Again, fix a C^*-dynamical system (A, α). We will say that a set $M \subseteq A$ is *α-invariant*, if $\alpha_t(M) \subseteq M$ for all $t \in \mathbb{R}$.

Corollary 2.1 (Theorem 1.3.7, Lemma 1.3.6, Proposition 1.3.9)**.** *The analytic elements form a dense, α-invariant *-subalgebra of A.*

Definition 2.2. For $0 \leq \beta \leq \infty$, a state ϕ of A is said to satisfy the *Kubo–Martin–Schwinger* (KMS) *condition at β for α* if:

($\beta = 0$:) ϕ is a trace which is *α-invariant*, i.e. $\forall x \in A, t \in \mathbb{R}: \quad \phi\big(\alpha_t(x)\big) = \phi(x).$

($0 < \beta < \infty$:) $\forall x \in A^a, y \in A, z \in \mathbb{C}: \quad \phi\big(y \cdot \alpha_{z+\mathbf{i}\beta}(x)\big) = \phi\big(\alpha_z(x) \cdot y\big).$

($\beta = \infty$:) $\forall x \in A^a, y \in A, z \in \mathbb{H}: \quad \big|\phi\big(y \cdot \alpha_z(x)\big)\big| \leq \|x\|\,\|y\|\,.$

For finite β, we will call such ϕ an *α-KMS$_\beta$-state,* or short *KMS$_\beta$*. For $\beta = \infty$, we say that ϕ is a *ground state.*

Theorem 2.3. *Let ϕ be a state of A. For fixed $0 < \beta < \infty$, the following are equivalent:*

(i) *ϕ is a KMS$_\beta$-state.*

(ii) *There is an α-invariant subset $M \subseteq A^a$ with dense span in A such that*

$$\phi\big(c \cdot \alpha_{\mathbf{i}\beta}(d)\big) = \phi(d \cdot c) \qquad \text{for all } c, d \in M. \tag{2}$$

(iii) *If we write $\Omega_\beta := \{z \in \mathbb{C} : 0 \leq \mathrm{Im}(z) \leq \beta\}$, then for all $x, y \in A$, there is a function $f \in \mathrm{C_b}(\Omega_\beta)$ which is analytic on $\mathring{\Omega}_\beta \subseteq \mathbb{C}$ such that*

$$\forall t \in \mathbb{R}: \quad f(t) = \phi\big(y \cdot \alpha_t(x)\big) \text{ and } f(t + \mathbf{i}\beta) = \phi\big(\alpha_t(x) \cdot y\big).$$

Proof.

ad (i) ⇒ (ii): Follows from Corollary 2.1.

ad (i) ⇒ (iii): Take $x, y \in A$ and a sequence $(x_n)_n$ in A^a converging to x. We know that for each n, the function $f_n\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \phi\big(y \cdot \alpha_z(x_n)\big)$, is entire and by assumption it satisfies

$$f_n(z + \mathbf{i}\beta) = \phi\big(\alpha_z(x_n) \cdot y\big) \tag{3}$$

for all $z \in \mathbb{C}$. We see that f_n is bounded on Ω_β: the function $[0, \beta] \to A$, $t \mapsto \alpha_{\mathbf{i}t}(x_n)$, is continuous and hence achieves its finite maximum m, so with Lemma 1.3.6 we get for an arbitrary $z = s + \mathbf{i}t$ with $0 \leq t \leq \beta$:

$$\begin{aligned} |f_n(z)| &= \left|\phi\big(y \cdot \alpha_{s+\mathbf{i}t}(x_n)\big)\right| \leq \|y \cdot \alpha_{s+\mathbf{i}t}(x_n)\| \\ &\leq \|y\| \left\|\alpha_s\big(\alpha_{\mathbf{i}t}(x_n)\big)\right\| = \|y\| \, \|\alpha_{\mathbf{i}t}(x_n)\| \leq \|y\| \, m < \infty. \end{aligned} \tag{4}$$

We can therefore apply Proposition 1.2.3 to the function $g := f_n - f_m$ to get for all $0 \leq t \leq \beta$:

$$\begin{aligned} \sup_{s \in \mathbb{R}} |g(s + \mathbf{i}t)| &\leq \sup_{s \in \mathbb{R}} |g(s)|^{\beta - t} \cdot \sup_{s \in \mathbb{R}} |g(s + \mathbf{i}\beta)|^t \\ &\leq \max\left(\sup_{s \in \mathbb{R}} |g(s)|^\beta ; \sup_{s \in \mathbb{R}} |g(s + \mathbf{i}\beta)|^\beta\right). \end{aligned}$$

Using the definition of g and Equation (3), this means:

$$\begin{aligned} \|f_n - f_m\|_{\Omega_\beta} &\leq \max\left(\sup_{s \in \mathbb{R}} \left|\phi\big(y \cdot \alpha_s(x_n - x_m)\big)\right| ; \sup_{s \in \mathbb{R}} \left|\phi\big(\alpha_s(x_n - x_m) \cdot y\big)\right|\right) \\ &\leq \|y\| \, \|x_n - x_m\| \longrightarrow 0, \end{aligned}$$

so the sequence $(f_n)_n$ converges uniformly to a function $f \in \mathrm{C}_b(\Omega_\beta)$. By Lemma 1.2.1, f is analytic on the interior of Ω_β. Moreover, for $t \in \mathbb{R}$, we have

$$\begin{aligned} \left|f(t + \mathbf{i}\beta) - \phi\big(\alpha_t(x) \cdot y\big)\right| &\leq |f(t + \mathbf{i}\beta) - f_n(t + \mathbf{i}\beta)| + \left|\phi\big(\alpha_t(x_n - x) \cdot y\big)\right| \\ &\leq |f(t + \mathbf{i}\beta) - f_n(t + \mathbf{i}\beta)| + \|x_n - x\| \, \|y\| \longrightarrow 0, \end{aligned}$$

so $f(t + \mathbf{i}\beta) = \phi\big(\alpha_t(x) \cdot y\big)$, and similarly $f(t) = \phi\big(y \cdot \alpha_t(x)\big)$.

ad (ii) ⇒ (i): In order to keep track of what we prove along the way, we will use the following convention for tags: for subsets $Y \subseteq \mathcal{M}(A)$, $X \subseteq A^a$, and $W, Z \subseteq \mathbb{C}$, let $[X, Y, Z, W]$ denote the following statement:

$$\forall x \in X, y \in Y, z \in Z, w \in W: \qquad \phi\big(y \cdot \alpha_{z+w}(x)\big) = \phi\big(\alpha_z(x) \cdot y\big).$$

Define $A_0 := \operatorname{span}(M)$. By linearity of the involved maps, Condition 2 also holds for $c, d \in A_0$. If $d \in A_0$, $y \in A$, and $(c_n)_n$ in A_0 converges to y, then:

$$\begin{aligned} \left\|\phi(d \cdot y) - \phi\big(y \cdot \alpha_{\mathbf{i}\beta}(d)\big)\right\| &\leq \|\phi(d \cdot y) - \phi(d \cdot c_n)\| + \left\|\phi(d \cdot c_n) - \phi\big(y \cdot \alpha_{\mathbf{i}\beta}(d)\big)\right\| \\ &\leq \|d \cdot y - d \cdot c_n\| + \left\|\phi\big(c_n \cdot \alpha_{\mathbf{i}\beta}(d)\big) - \phi\big(y \cdot \alpha_{\mathbf{i}\beta}(d)\big)\right\| \qquad \text{as } c_n \in A_0 \\ &\leq \|d\| \, \|y - c_n\| + \|c_n - y\| \, \left\|\alpha_{\mathbf{i}\beta}(d)\right\| \longrightarrow 0. \end{aligned}$$

As $\alpha_t(A_0) \subseteq A_0$ for $t \in \mathbb{R}$, this implies for $y \in A$ and $d \in A_0$,

$$\phi\big(y \cdot \alpha_{t+\mathbf{i}\beta}(d)\big) = \phi\big(\alpha_t(d) \cdot y\big). \qquad [A_0, A, \mathbb{R}, \{\mathbf{i}\beta\}]$$

Now fix $d \in A_0$ and an approximate unit $(u_\mu)_\mu$ of A. Since $\phi\big(u_\mu \cdot \alpha_{\mathbf{i}\beta}(d)\big) = \phi\big(d \cdot u_\mu\big)$, we infer

$$\begin{aligned} \big|\phi\big(\alpha_{\mathbf{i}\beta}(d)\big) - \phi(d)\big| &\leq \big|\phi\big(\alpha_{\mathbf{i}\beta}(d)\big) - \phi\big(u_\mu \cdot \alpha_{\mathbf{i}\beta}(d)\big)\big| + \big|\phi\big(u_\mu \cdot \alpha_{\mathbf{i}\beta}(d)\big) - \phi(d)\big| \\ &\leq \big\|\alpha_{\mathbf{i}\beta}(d) - u_\mu \cdot \alpha_{\mathbf{i}\beta}(d)\big\| + \big|\phi\big(d \cdot u_\mu - d\big)\big| \\ &\leq \big\|\alpha_{\mathbf{i}\beta}(d) - u_\mu \cdot \alpha_{\mathbf{i}\beta}(d)\big\| + \|d \cdot u_\mu - d\| \longrightarrow 0. \end{aligned}$$

Hence, $\phi\big(\alpha_{\mathbf{i}\beta}(d)\big) = \phi(d)$. Again, since $\alpha_t(A_0) \subseteq A_0$, it follows with Lemma 1.3.6:

$$\phi\big(\alpha_{t+\mathbf{i}\beta}(d)\big) = \phi\big(\alpha_{\mathbf{i}\beta}\big(\alpha_t(d)\big)\big) = \phi\big(\alpha_t(d)\big). \qquad [A_0, 1, \mathbb{R}, \{\mathbf{i}\beta\}]$$

Thus, the entire function $f : w \mapsto \phi\big(\alpha_w(d)\big)$ agrees with $w \mapsto \phi\big(\alpha_{w+\mathbf{i}\beta}(d)\big)$ on the real line and therefore everywhere, i.e. f has periodicity $\mathbf{i}\beta$. Since $[0,\beta] \to A$, $t \mapsto \alpha_{\mathbf{i}t}(d)$, is continuous and hence achieves its finite maximum m, we get for any $w = s + \mathbf{i}t$ with $t \in [0,\beta]$:

$$|f(w)| = \big|\phi\big(\alpha_{s+\mathbf{i}t}(d)\big)\big| \leq \|\alpha_{s+\mathbf{i}t}(d)\| = \big\|\alpha_s\big(\alpha_{\mathbf{i}t}(d)\big)\big\| = \|\alpha_{\mathbf{i}t}(d)\| \leq m.$$

But, because of its periodicity, this implies that f is bounded everywhere, hence constant by Liouville's Theorem. To sum up, we have shown that for all $d \in A_0$ and $w \in \mathbb{C}$:

$$\phi\big(\alpha_w(d)\big) = \phi(d). \qquad [A_0, 1, \{0\}, \mathbb{C}]$$

Now, for $t \in \mathbb{R}$ and an arbitrary $x \in A$ with $(d_n)_n$ in A_0 converging to it, we get

$$\begin{aligned} \big|\phi\big(\alpha_t(x)\big) - \phi(x)\big| &\leq \big|\phi\big(\alpha_t(x)\big) - \phi\big(\alpha_t(d_n)\big)\big| + \big|\phi\big(\alpha_t(d_n)\big) - \phi(x)\big| \\ &\leq \|\alpha_t(x) - \alpha_t(d_n)\| + |\phi(d_n) - \phi(x)| \qquad \text{by } [A_0, 1, \{0\}, \mathbb{C}] \\ &\leq 2\,\|d_n - x\| \longrightarrow 0, \end{aligned}$$

so $\phi\big(\alpha_t(x)\big) = \phi(x)$, and we have shown that ϕ is α-invariant. In particular, for $x \in A^a$, the entire function $w \mapsto \phi\big(\alpha_w(x)\big)$ is identically $\phi(x)$ on $\mathbb{R}$ and hence everywhere, i.e. for all $w \in \mathbb{C}$,

$$\phi\big(\alpha_w(x)\big) = \phi(x). \qquad [A^a, 1, \{0\}, \mathbb{C}]$$

For $y' \in A$ and $d \in A_0$, the entire functions $z \mapsto \phi\big(y' \cdot \alpha_{z+\mathbf{i}\beta}(d)\big)$ and $z \mapsto \phi\big(\alpha_z(d) \cdot y'\big)$ agree on the real line because of $[A_0, A, \mathbb{R}, \{\mathbf{i}\beta\}]$ and hence everywhere, i.e. for all $z \in \mathbb{C}$,

$$\phi\big(y' \cdot \alpha_{z+\mathbf{i}\beta}(d)\big) = \phi\big(\alpha_z(d) \cdot y'\big). \qquad [A_0, A, \mathbb{C}, \{\mathbf{i}\beta\}]$$

(Note that it is not immediately clear why this should hold more generally for A^a instead of its subset A_0!) For $y \in A^a$, we apply the above line to get

$$\begin{aligned}\phi\big(\alpha_{-\mathbf{i}\beta}(y)\cdot d\big) &= \phi\big(\alpha_{-\mathbf{i}\beta}(y)\cdot\alpha_{-\mathbf{i}\beta+\mathbf{i}\beta}(d)\big) && \text{as } \alpha_0 = \mathrm{id}\\ &= \phi\big(\alpha_{-\mathbf{i}\beta}(d)\cdot\alpha_{-\mathbf{i}\beta}(y)\big) && \text{by } [A_0, A, \mathbb{C}, \{\mathbf{i}\beta\}] \text{ for } y' := \alpha_{-\mathbf{i}\beta}(y),\ z = -\mathbf{i}\beta\\ &= \phi\big(\alpha_{-\mathbf{i}\beta}(d\cdot y)\big) = \phi(d\cdot y) && \text{by } [A^a, 1, \{0\}, \mathbb{C}].\end{aligned}$$

Therefore, if $x, y \in A^a$ and $d_n \in A_0$ converging to x, we get

$$\begin{aligned}\phi\big(y\cdot\alpha_{\mathbf{i}\beta}(x)\big) &= \phi\Big(\alpha_{\mathbf{i}\beta}\big(\alpha_{-\mathbf{i}\beta}(y)\cdot x\big)\Big) = \phi\big(\alpha_{-\mathbf{i}\beta}(y)\cdot x\big) \qquad \text{by } [A^a, 1, \{0\}, \mathbb{C}]\\ &= \lim_n \phi\big(\alpha_{-\mathbf{i}\beta}(y)\cdot d_n\big) = \lim_n \phi(d_n\cdot y) = \phi(x\cdot y).\end{aligned}$$

By replacing x with $\alpha_z(x)$ and approximating a more general $y \in A$ by $y_n \in A^a$, we can use this to get

$$\phi\big(y\cdot\alpha_{z+\mathbf{i}\beta}(x)\big) = \phi\Big(y\cdot\alpha_{\mathbf{i}\beta}\big(\alpha_z(x)\big)\Big) = \phi\big(\alpha_z(x)\cdot y\big), \qquad \text{i.e. } [A^a, A, \mathbb{C}, \{\mathbf{i}\beta\}].$$

ad (iii) ⇒ (i): For $x \in A^a$ and $y \in A$, the functions $g, h : \mathbb{C} \to \mathbb{C}$ given by $g(z) := \phi\big(y\cdot\alpha_z(x)\big)$ and $h(z) := \phi\big(\alpha_{z-\mathbf{i}\beta}(x)\cdot y\big)$, are well-defined and entire. We want to show that these functions are identical. For our fixed x and y, we are given the function f as in (iii), so the functions $g - f$ and $h - f$ are in $\mathrm{C}(\Omega_\beta)$, analytic on $\mathring{\Omega}_\beta$, and satisfy

$$(g-f)|_{\mathbb{R}} \equiv 0 \quad \text{and} \quad (h-f)|_{\mathbb{R}+\mathbf{i}\beta} \equiv 0.$$

By Corollary 1.2.6, we infer that $g - f \equiv 0$ on $\Omega_\beta \smallsetminus (\mathbb{R} + \mathbf{i}\beta)$ and $h - f \equiv 0$ on $\Omega_\beta \smallsetminus \mathbb{R}$, so in particular $g \equiv h$ on $\mathring{\Omega}_\beta$. As the functions are entire, they are identical everywhere by the Identity Theorem. □

Remark 2.4. In (ii), we could equivalently assume ϕ instead of M to be α-invariant: one again shows that Equation (2) still holds for any $c \in A$ and $d \in \mathrm{span}(M) =: A_0$. If we then take $t \in \mathbb{R}$, we get

$$\begin{aligned}\phi\Big(c\cdot\alpha_{\mathbf{i}\beta}\big(\alpha_t(d)\big)\Big) &\overset{\alpha\text{-inv.}}{=} \phi\Big(\alpha_{-t}\Big(c\cdot\alpha_{\mathbf{i}\beta}\big(\alpha_t(d)\big)\Big)\Big) \overset{1.3.6}{=} \phi\Big(\alpha_{-t}(c)\cdot\alpha_{\mathbf{i}\beta}(d)\Big)\\ &\overset{d\in A_0}{=} \phi\Big(d\cdot\alpha_{-t}(c)\Big) \overset{\alpha\text{-inv.}}{=} \phi\Big(\alpha_t\big(d\cdot\alpha_{-t}(c)\big)\Big) = \phi\big(\alpha_t(d)\cdot c\big).\end{aligned}$$

Therefore, the α-invariant set $\bigcup_{t\in\mathbb{R}} \alpha_t(M)$ also satisfies Equation (2).

We also have the following very similar equivalences in the infinite case:

Theorem 2.5. *For a state ϕ of A, the following are equivalent:*

(i) *ϕ is a ground state.*

(ii) *ϕ satisfies the KMS-condition at infinity for elements in a dense subset $M \subseteq A^a$.*

(iii) *For all* $x, y \in A$, *there is a function* $f \in \mathrm{C_b}(\mathbb{H})$ *which is analytic on* $\mathring{\mathbb{H}}$ *such that*

$$\forall t \in \mathbb{R}: \quad f(t) = \phi\big(y \cdot \alpha_t(x)\big), \ \textit{and} \ \|f\|_{\mathbb{H}} \le \|x\| \, \|y\| \, .$$

Proof.

ad (i) ⇒ (ii): Clear.

ad (ii) ⇒ (iii): Fix $y \in A$, and let $(y_n)_n$ be a sequence in M converging to y. If $x_0 \in M$, then the condition on ϕ gives for all $z \in \mathbb{C}$,

$$\begin{aligned}\big|\phi\big(y \cdot \alpha_z(x_0)\big)\big| &\le \big|\phi\big(y \cdot \alpha_z(x_0)\big) - \phi\big(y_n \cdot \alpha_z(x_0)\big)\big| + \big|\phi\big(y_n \cdot \alpha_z(x_0)\big)\big| \\ &\le \|y - y_n\| \cdot \|\alpha_z(x_0)\| + \|y_n\| \cdot \|x_0\| \overset{n\to\infty}{\longrightarrow} \|y\| \cdot \|x_0\| \, ,\end{aligned}$$

so the inequality holds for arbitrary y, too. Thus, we now only have to generalize to arbitrary analytic x: Take a sequence $(x_n)_n$ in M converging to x. We know that for each n, the function $f_n \colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \phi\big(y \cdot \alpha_z(x_n)\big)$, is entire and satisfies

$$|f_n(z)| \le \|x_n\| \, \|y\| \quad \text{for } z \in \mathbb{H}. \tag{5}$$

Let us without loss of generality assume that $y \neq 0$. For an arbitrary $\epsilon > 0$, we define

$$G(z) := G^{\epsilon}_{n,m}(z) := \frac{\big(f_n - f_m\big)(z)}{\|y\| \cdot \big(\|x_n - x_m\| + \epsilon\big)}, \quad z \in \mathbb{C}.$$

For this entire function, we have on $\partial\mathbb{H} = \mathbb{R}$:

$$|G(t)| = \frac{\big|\phi\big(y \cdot \alpha_t(x_n - x_m)\big)\big|}{\|y\| \cdot \big(\|x_n - x_m\| + \epsilon\big)} \le \frac{\|y\| \cdot \|x_n - x_m\|}{\|y\| \cdot \big(\|x_n - x_m\| + \epsilon\big)} < 1,$$

and we have for any $z \in \mathbb{H}$:

$$|G(z)| \le \frac{|f_n(z)| + |f_m(z)|}{\|y\| \cdot \big(\|x_n - x_m\| + \epsilon\big)} \le \frac{\|y\| \left(\|x_n\| + \|x_m\|\right)}{\|y\| \cdot \big(\|x_n - x_m\| + \epsilon\big)},$$

where the last inequality is due to Equation (5). Since the right-hand side is independent of $\arg(z)$, we can apply Proposition 1.2.4 to get:

$$\forall z \in \mathbb{H}: \quad |G(z)| \le 1,$$

in other words:

$$\forall z \in \mathbb{H}: \quad |f_n(z) - f_m(z)| \le \|y\| \left(\|x_n - x_m\| + \epsilon\right).$$

Since ϵ was arbitrary, we get

$$\|f_n - f_m\|_{\mathbb{H}} \le \|y\| \, \|x_n - x_m\| \longrightarrow 0.$$

so the sequence $(f_n)_n$ converges uniformly to a function $f \in C_b(\Omega_\beta)$ which is analytic on the interior of $\mathbb{H}$ by Lemma 1.2.1 and has the asserted properties.

ad (iii) ⇒ (i): Taking x, y, g as in the proof for $\beta < \infty$, we are again given a function f as in (iii). The same argument now yields that $g \equiv f$ on all of $\mathbb{H}$, so in particular we get for $z \in \mathbb{H}$ that

$$\left|\phi\big(y \cdot \alpha_z(x)\big)\right| \overset{\text{def}}{=} |g(z)| = |f(z)| \leq \|x\| \, \|y\| . \qquad \square$$

Corollary 2.6. *Any α-KMS-state ϕ is α-invariant.*

For $\beta = 0$, the statement is true by definition, and we have also already seen why it holds for finite β on page 9.

Proof for ground states. As argued before, it suffices to show the claim for analytic elements, so take $x \in A^a$. Since ϕ is a ground state, we can use an approximate unit $(u_\mu)_\mu$ of A of norm 1 to see that the map $f\colon z \mapsto \phi\big(\alpha_z(x)\big)$ satisfies

$$|f(z)| \leq \|x\| \quad \text{for all } z \in \mathbb{H}.$$

Because of Corollary 2.1, we can infer the same for x^*, that is, the function $g\colon z \mapsto \phi\big(\alpha_z(x^*)\big) = \overline{\phi\big(\alpha_{\overline{z}}(x)\big)}$ satisfies

$$|g(z)| \leq \|x^*\| = \|x\| \quad \text{for all } z \in \mathbb{H}.$$

Since for any $z \in \mathbb{C}$,

$$|f(z)|^2 = f(z)\overline{f(z)} = \overline{g(\overline{z})}g(\overline{z}) = |g(\overline{z})|^2 ,$$

both inequalities together yield that f is bounded everywhere by $\|x\|$. The function is thus constant by Liouville's Theorem, and we are done. $\square$

3 A FINITE DIMENSIONAL EXAMPLE

We are now going to discuss parts of the example given in [6], Section 1.3. The C^*-algebra in question is $\mathfrak{A} := \mathrm{M}_n(\mathbb{C})$. Defining

$$\alpha_t(A) := \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH} \quad \text{for } t \in \mathbb{R}, A \in \mathfrak{A}, \tag{6}$$

for H a fixed self-adjoint matrix, obviously gives us an $\mathbb{R}$-action on $\mathfrak{A}$. Conversely, using Stone's Theorem (cf. [8], Theorem 7.1.7) and the fact that all automorphisms of $\mathfrak{A}$ are inner (see [8], Lemma 8.7.4), any C^*-dynamical system $(\mathfrak{A}, \alpha)$ arises this way.[2] Note that every $A \in \mathfrak{A}$ is α-analytic since Equation (6) can be extended to $\mathbb{C}$ simply by replacing t with z (modulo knowing that this is still well-defined and well-behaved).[3] Let us first understand arbitrary states on $\mathfrak{A}$. Recall that matrix $Q \in \mathfrak{A}$ is positive as element of $\mathfrak{A}$ if and only if it is self-adjoint and has only non-negative eigenvalues.

2 The matrix H is a so-called *Hamiltonian*, and it is determined by α up to an additive constant.
3 Don't forget that α_z need not be an automorphism of $\mathfrak{A}$.

Definition 3.1. We say that Q is a *density matrix* if $Q \geq 0$ and $\mathrm{Tr}(Q) = 1$.

Theorem 3.2. *The positive linear functionals on $\mathfrak{A}$ are in one-to-one correspondence with positive matrices, the states correspond to density matrix, and pure states to rank-one projections.*

Lemma 3.3. *For every matrix Q, there is a unitary matrix U such that $\tilde{Q} := U^*QU$ is an upper triangular matrix. Any such arising $\tilde{Q}$ has the eigenvalues of Q on its diagonal and, if Q is self-adjoint, then $\tilde{Q}$ is a diagonal matrix.*

Proof. Existence of such a U is shown in [10], Chapter 6, Theorem 1.1, and the part about the eigenvalues in [1], Proposition 5.18. If Q is self-adjoint, then so is $\tilde{Q}$. As an upper triangular matrix, $\tilde{Q}$ does not have any entries below the diagonal, so if it is self-adjoint, $\tilde{Q}$ can only have diagonal entries. □

Proof of Theorem 3.2. Fix a density matrix Q and define

$$\phi_Q \colon \mathfrak{A} \to \mathbb{C}, \quad A \mapsto \mathrm{Tr}(AQ).$$

This linear functional is positive: since $Q \geq 0$, we have $AQA^* \geq 0$ for any A, which implies that all eigenvalues of AQA^* are non-negative. Hence, their sum is non-negative, so $\phi_Q(A^*A) \geq 0$. Moreover, since

$$\left\|\phi_Q\right\|_{\mathrm{op}} = \phi_Q(1_{\mathfrak{A}}), \tag{7}$$

we get $\left\|\phi_Q\right\| = 1$ as $\mathrm{Tr}(Q) = 1$, so ϕ_Q indeed defines a state.

Conversely, let ϕ be any state. By Riesz Representation Theorem[4] (cf. [4], Theorem 3.4), we get a unique Q_ϕ in $\mathfrak{A}$ such that

$$\mathrm{Tr}\left(AQ_\phi\right) = \phi(A) \quad \text{for all } A \in \mathfrak{A}. \tag{8}$$

Because of Equation (7), we get $\mathrm{Tr}\left(Q_\phi\right) = 1$. For all A, we have

$$\mathrm{Tr}\left(AQ_\phi^*\right) = \mathrm{Tr}\left(\left(Q_\phi A^*\right)^*\right) = \overline{\mathrm{Tr}\left(Q_\phi A^*\right)} = \overline{\phi(A^*)} = \phi(A),$$

where the last equation holds because ϕ is positive, so in particular self-adjoint. Since Q_ϕ was unique with Property (8), we get that Q_ϕ is self-adjoint. Now let U, $\tilde{Q}_\phi$ be as in Lemma 3.3. If we denote by E_{ij} the matrix with only one 1 in the i^{th} row and the j^{th} column and zeros everywhere else, then by the positivity of ϕ we get

$$\begin{aligned} 0 &\leq \phi\big((UE_{ii})(UE_{ii})^*\big) = \phi(UE_{ii}U^*) = \mathrm{Tr}\left(\left(UE_{ii}U^*\right)Q_\phi\right) \\ &= \mathrm{Tr}\left(E_{ii}\tilde{Q}_\phi\right) = \mathrm{i}^{\mathrm{th}} \text{ diagonal entry of } \tilde{Q}_\phi, \end{aligned}$$

so the self-adjoint matrix Q_ϕ has only non-negative eigenvalues, that is, Q_ϕ is positive semi-definite. We have thus shown that Q_ϕ is a density matrix.

4 Here, we use that $\mathfrak{A}$ is a Hilbert space with respect to the Frobenius inner product defined by $\langle A \,|\, B\rangle := \mathrm{Tr}(AB^*)$, where B^* is the conjugate transpose.

Now for the pure resp. rank-one projection part:

Note that a rank-one density matrix Q is automatically a projection: since Q is self-adjoint, $\tilde{Q}$ is diagonal, and as $\tilde{Q}$ has the same rank and trace as Q, it is just E_{kk} for some k. Hence, $Q = U\tilde{Q}U^* = U\tilde{Q}^2U^* = U\tilde{Q}U^*U\tilde{Q}U^* = Q^2$, so Q is a projection.

Moreover, since $A \mapsto UAU^*$ is an automorphism of $\mathfrak{A}$, we know that ϕ_Q is pure if and only if $\phi_{\tilde{Q}}$ is, and that Q is rank-one if and only of $\tilde{Q}$ is. Thus, we can without loss of generality assume that $Q = \tilde{Q}$ is a diagonal matrix.

To show that pure states have rank-one density matrices, let us assume that Q is not rank-one. Let λ be one of the (at least two) non-zero entries in the, say, k^{th} diagonal position. Since $Q \geq 0$ and $\operatorname{Tr}(Q) = 1$, we have $0 < \lambda < 1$. The matrices $R := E_{kk}$ and $P := \frac{1}{1-\lambda}(Q - \lambda R)$ are again density matrices and

$$Q = \lambda R + (1-\lambda)P,$$

so ϕ_Q cannot be pure.

Showing that rank-one matrices give pure states is left as an exercise.[5] □

Definition 3.4. For a fixed C^*-dynamical system $(\mathfrak{A}, \alpha)$ with H as in Equation (6) and for $0 < \beta < \infty$, we define the *Gibbs state* as the state ϕ_G corresponding to the density matrix

$$Q_G := \frac{1}{\operatorname{Tr}\left(\mathrm{e}^{-\beta H}\right)}\mathrm{e}^{-\beta H}.$$

Theorem 3.5. *The Gibbs state is the unique KMS$_\beta$-state.*

Lemma 3.6. *If $X \in \mathfrak{A}$ is such that $\operatorname{Tr}(AX) = 0$ for all $A \in \mathfrak{A}$, then $X = 0$.*

Proof. Follows from plugging in E_{ij} for all $1 \leq i, j \leq n$ since $0 = \operatorname{Tr}(E_{ij}X) = x_{ij}$ gives the $i-j$-entry of X. □

Proof of Theorem 3.5. For the Gibbs state ϕ_G to be KMS$_\beta$, we need that

$$\phi_G\big(A\alpha_{z+i\beta}(B)\big) = \phi_G\big(\alpha_z(B)A\big)$$

for all $A, B \in \mathfrak{A}$ and $z \in \mathbb{C}$. Using the trace property, we get

$$\begin{aligned}\operatorname{Tr}\Big(A\left(\mathrm{e}^{\mathbf{i}(z+i\beta)H}B\mathrm{e}^{-\mathbf{i}(z+i\beta)H}\right)\mathrm{e}^{-\beta H}\Big) &= \operatorname{Tr}\Big(A\mathrm{e}^{-\beta H}(\mathrm{e}^{\mathbf{i}zH}B\mathrm{e}^{-\mathbf{i}zH})\mathrm{e}^{\beta H}\mathrm{e}^{-\beta H}\Big) \\ &= \operatorname{Tr}\Big(\left(\mathrm{e}^{\mathbf{i}zH}B\mathrm{e}^{-\mathbf{i}zH}\right)A\mathrm{e}^{-\beta H}\Big) \quad \text{since } \mathrm{e}^{\beta H}\mathrm{e}^{-\beta H} = \mathrm{e}^{0H} = 1.\end{aligned}$$

By the definitions of Q_G, α, and ϕ_G, this yields the claim.

Let us now assume that ϕ is any KMS$_\beta$-state and write Q for its density matrix. By Corollary 2.6, we know that ϕ is α-invariant, that is:

$$\forall t \in \mathbb{R}, A \in \mathfrak{A}: \quad \operatorname{Tr}(AQ) = \operatorname{Tr}\left(\mathrm{e}^{\mathbf{i}tH}A\mathrm{e}^{-\mathbf{i}tH}Q\right) = \operatorname{Tr}\left(A\mathrm{e}^{-\mathbf{i}tH}Q\mathrm{e}^{\mathbf{i}tH}\right),$$

5 Part of this exercise is reporting back to me if you happen to find a Linear Algebra proof, i.e. one that does not invoke an equivalent of [3], Proposition II.6.4.8.

where the second equality is due to the trace property. Lemma 3.6 gives $Q = \mathrm{e}^{-\mathbf{i}tH} Q \mathrm{e}^{\mathbf{i}tH}$ for all $t \in \mathbb{R}$. Therefore, the analytic function $z \mapsto Q - \mathrm{e}^{-\mathbf{i}zH} Q \mathrm{e}^{\mathbf{i}zH}$ is zero on the real line, so by the Identity Theorem, it is zero everywhere. We have shown:

$$\forall z \in \mathbb{C} : \quad \mathrm{e}^{\mathbf{i}zH} Q = Q \mathrm{e}^{\mathbf{i}zH}. \tag{9}$$

Since ϕ is KMS_β, we know that

$$\begin{aligned} \forall A, B \in \mathfrak{A} : \quad \mathrm{Tr}\left(BAQ\right) &= \mathrm{Tr}\left(A\mathrm{e}^{-\beta H} B \mathrm{e}^{\beta H} Q\right) \\ &= \mathrm{Tr}\left(B\mathrm{e}^{\beta H} Q A \mathrm{e}^{-\beta H}\right) \text{ by the trace property.} \end{aligned}$$

Because of Lemma 3.6, we now get

$$\forall A \in \mathfrak{A} : \quad A\left(Q\mathrm{e}^{\beta H}\right) = \left(\mathrm{e}^{\beta H} Q\right) A \overset{(9)}{=} \left(Q\mathrm{e}^{\beta H}\right) A,$$

that is, $Q\mathrm{e}^{\beta H}$ commutes with every element of $\mathfrak{A}$. It is thus a multiple of the identity: $Q\mathrm{e}^{\beta H} = \lambda \mathrm{Id}$. Since Q has trace 1 as a density matrix, we conclude $Q = Q_G$, and so ϕ is the Gibbs state. □

REFERENCES

[1] Sheldon Axler. *Linear algebra done right*. Second. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1997, pp. xvi+251.

[2] Erik P. van den Ban. *Tools from Lebesgue integration*. 2005. URL: http://www.staff.science.uu.nl/~ban00101/dist2005/integration.pdf (visited on 05/17/2015).

[3] Bruce Blackadar. *Operator algebras. Theory of C^*-algebras and von Neumann algebras*. July 19, 2013. URL: http://wolfweb.unr.edu/homepage/bruceb/Cycr.pdf (visited on 12/04/2015).

[4] John B. Conway. *A course in functional analysis*. Second. Vol. 96. Graduate Texts in Mathematics. Springer-Verlag, New York, 1990, pp. xvi+399.

[5] J. M. G. Fell and R. S. Doran. *Representations of *-algebras, locally compact groups, and Banach *-algebraic bundles. Vol. 1*. Vol. 125. Pure and Applied Mathematics. Basic representation theory of groups and algebras. Academic Press, Inc., Boston, MA, 1988, pp. xviii+746.

[6] N. M. Hugenholtz. "C^*-algebras and statistical mechanics". In: *Operator algebras and applications, Part 2 (Kingston, Ont., 1980)*. Vol. 38. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, R.I., 1982, pp. 407–465.

[7] Gert K. Pedersen. *Analysis now*. Vol. 118. Graduate Texts in Mathematics. Springer-Verlag, New York, 1989, pp. xiv+277.

[8] Gert K. Pedersen. *C^*-algebras and their automorphism groups*. Vol. 14. London Mathematical Society Monographs. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979, pp. ix+416.

[9] Walter Rudin. *Real and complex analysis*. Third. McGraw-Hill Book Co., New York, 1987, pp. xiv+416.

[10] Sergei Treil. *Linear algebra done wrong*. Sept. 2014. URL: http://www.math.brown.edu/~treil/papers/LADW/LADW-2014-09.pdf (visited on 05/17/2015).

[11] František Wolf. "An extension of the Phragmén-Lindelöf theorem". In: *J. London Math. Soc.* 14 (1939), pp. 208–216.